Generalized Phase-Space Representation of Operators

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Introducing asymmetry into the Weyl representation of operators leads to a variety of phase-space representations and new symbols. Specific generalizations of the Husimi and the Glauber-Sudarshan symbols are explicitly derived.

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I. INTRODUCTION

Phase space representations have proven their value in classical mechanics and ever since the work of Wigner [1] in quantum mechanics as well. The map introduced by Wigner from state functions to quasi-probability functions is notable for two fundamental properties. The first reason it is notable relates to the fact that, in general, the Wigner quasi-probability function is not everywhere nonnegative, and this keeps it from being a true probability function. The second reason it is notable relates to the fact that the Wigner map is not unique, and this property has led to a variety of alternative prescriptions to define a quantum mechanical phase space distribution. The most widely known variation on the Wigner map is that due to Cohen [2]. Cohen's generalization, which is typical of most such efforts, normally steps outside the family of operator images by the Wigner map; in other words, generalizations are introduced that involve an auxiliary phase-space function that is generally not obtained by means of an operator map.

Two well known phase-space representations for operators, however, are defined by direct transcriptions of operators. In particular, we refer to the Husimi phase-space representation, and its dual, the Glauber-Sudarshan phase-space representation, both of which are recalled below. Both representations can be defined in an entirely abstract operator setting with the only transcription between operators and phase space being given by an exclusive use of the Weyl operator. The goal of the present paper is the generalization of such schemes leading to a multitude of such dual pairs that are defined in a strictly operator fashion, and which, thereby, transform covariantly under coordinate transformations.

While our discussion is presented in the notation and language of quantum mechanics, a completely parallel discussion applies to a time-frequency analysis of signals and their transformations; in this connection, see [3], Sec. 4.5 and Chap. 9.

A. Basics: Wigner, Weyl, and Moyal

The basic phase space representation of quantum states and operators is due to Wigner [1], Weyl [4] and Moyal [5]. The basis of this representation involves the Heisenberg canonical operator pair, P and Q, subject to the commutation rule $[Q,P]=i$ (with $\hbar=1$ throughout), expressed in their exponential form

$$U[p,q]\equiv e^{i(pQ-qP)}\ , \tag{1}$$

and known as the Weyl operator. As such, we assume the operators Q and P are both self adjoint, with continuous spectrum on the entire real line, and thus the operators $U[p,q]$ are unitary and (weakly) continuous in the real parameters $(p,q)\in\mathbb{R}^2$. In terms of these operators the commutation relation assumes the form

$$\begin{aligned} U[p,q]\,U[r,s] &= e^{i(ps-qr)/2}\,U[p+r,q+s] \\ &= e^{i(ps-qr)}\,U[r,s]\,U[p,q]\ , \end{aligned} \tag{2}$$

which is the version we shall use. A key relation we shall employ is the distributional identity

$$\begin{aligned} \mathrm{Tr}\,(U[p,q]\,U[r,s]^\dagger) &= \mathrm{Tr}\,(U[r,s]^\dagger\,U[p,q]) \\ &= 2\pi\,\delta(p-r)\,\delta(q-s)\ , \end{aligned} \tag{3}$$

and it is appropriate that we reestablish this well known expression. To that end, consider the diagonal position space matrix elements

$$\begin{aligned} &\langle y|\,U[p,q]\,U[r,s]^\dagger\,|y\rangle \\ &= \langle y|\,e^{ipQ/2}e^{-iqP}\,e^{i(p-r)Q/2}\,e^{isP}\,e^{-irQ/2}\,|y\rangle \\ &= e^{i(p-r)y/2}\langle y|\,e^{-iqP}\,e^{i(p-r)Q/2}\,e^{isP}\,|y\rangle \\ &= e^{i(p-r)y/2}\langle y+q|\,e^{i(p-r)Q/2}\,|y+s\rangle \\ &= e^{i(p-r)y/2}\,e^{i(p-r)(y+s)/2}\,\delta(q-s)\ . \end{aligned} \tag{4}$$

Integration over y leads to

$$\begin{aligned} \mathrm{Tr}\,(U[p,q]\,U[r,s]^\dagger) &= \int e^{i(p-r)y}\,e^{i(p-r)s/2}\,\delta(q-s)\,dy \\ &= 2\pi\,\delta(p-r)\,\delta(q-s)\ , \end{aligned} \tag{5}$$

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as required. We make occasional use of this identity, which we call the *basic identity.*

The Weyl representation of operators (limited to Hilbert-Schmidt operators for the time being), is given by

$$A = \int U[k,x]\,\tilde{A}[k,x]\,\frac{dk\,dx}{2\pi}\;, \tag{6}$$

where $\tilde{A}[k,x]$ is a function to be determined. If we consider $\mathrm{Tr}\,(U[k',x']^\dagger\,A)$ and use the basic identity Eq. (5), we learn that

$$\begin{aligned}\mathrm{Tr}\,(U[k',x']^\dagger\,A) \;&=\; \int \mathrm{Tr}\,(U[k',x']^\dagger\,U[k,x])\,\tilde{A}[k,x]\frac{dk\,dx}{2\pi}\\ &=\; \tilde{A}[k',x']\;. \end{aligned} \tag{7}$$

Therefore, we have the important completeness relation

$$A = \int U[k,x]\,\mathrm{Tr}\,(U[k,x]^\dagger\,A)\,\frac{dk\,dx}{2\pi}\;. \tag{8}$$

We take a second operator with an analogous representation,

$$B = \int U[k,x]\,\mathrm{Tr}\,(U[k,x]^\dagger\,B)\,\frac{dk\,dx}{2\pi}\;, \tag{9}$$

and we again introduce the notation that

$$\tilde{B}[k,x] \equiv \mathrm{Tr}\,(U[k,x]^\dagger\,B)\;. \tag{10}$$

Using the basic identity Eq.(5) again, we learn that

$$\begin{aligned}&\mathrm{Tr}\,(A^\dagger\,B)\\ &= \int \tilde{A}[k',x']^*\mathrm{Tr}\,(U[k',x']^\dagger\,U[k,x])\tilde{B}[k,x]\,\frac{dk'\,dx'\,dk\,dx}{(2\pi)^2}\\ &= \int \tilde{A}[k,x]^*\,\tilde{B}[k,x]\,\frac{dk\,dx}{2\pi}\;. \end{aligned} \tag{11}$$

In summary, the Weyl representation for operators is given by

$$A = \int U[k,x]\,\tilde{A}[k,x]\,\frac{dk\,dx}{2\pi}\;, \tag{12}$$

where

$$\tilde{A}[k,x] \equiv \mathrm{Tr}\,(U[k,x]^\dagger\,A)\;. \tag{13}$$

These relations are complemented by the expression

$$\mathrm{Tr}\,(A^\dagger\,B) = \int \tilde{A}[k,x]^*\,\tilde{B}[k,x]\,\frac{dk\,dx}{2\pi}\;. \tag{14}$$

It will also be important to introduce the Fourier transforms of the Weyl representation elements as

$$A[p,q] \equiv \int e^{i(kq-xp)}\,\tilde{A}[k,x]\,\frac{dk\,dx}{2\pi}\;, \tag{15}$$

and, correspondingly,

$$B[p,q] \equiv \int e^{i(kq-xp)}\,\tilde{B}[k,x]\,\frac{dk\,dx}{2\pi}\;. \tag{16}$$

In terms of these representatives we have

$$\mathrm{Tr}\,(A^\dagger\,B) = \int A[p,q]^*\,B[p,q]\,\frac{dp\,dq}{2\pi}\;, \tag{17}$$

as follows from Parseval's Theorem.

We note that $A[p,q]$ and $B[p,q]$ as introduced here, are generally referred to as the Weyl symbols for the operators A and B, respectively. Hence, $\tilde{A}[k,x]$ and $\tilde{B}[k,x]$ are the Fourier transform of the corresponding Weyl symbols.

II. INITIAL INTRODUCTION OF ASYMMETRY

It is clear from Eqs. (11) and (17) that the Weyl representation has led to a *symmetric* functional realization in how it treats the operators A and B. This is a natural representation choice for applications in which A and B enter in a symmetric manner. However, there are cases when that is not appropriate, and in such situations it can prove valuable to treat the *representations* for A and B in an *asymmetric* manner. For example, it is well known that in quantum optics observables are commonly represented by normally ordered operators, which are readily given a phase space representation via the Husimi representation [6]. In turn, the dual representation for this important example is known as the Glauber-Sudarshan representation [7]. We begin our analysis of asymmetric representations with this important conjugate pair.

To discuss these alternative representations it is convenient to first recall canonical coherent states given by

$$|p,q\rangle \equiv U[p,q]\,|0\rangle\;, \tag{18}$$

where the fiducial vector $|0\rangle$ is the normalized ground state of a suitable harmonic oscillator, or more simply that

$$(Q+iP)\,|0\rangle = 0\;. \tag{19}$$

From a notational point of view, it is also useful to recall that

$$\begin{aligned}|p,q\rangle \;&\equiv\; |z\rangle = e^{za^\dagger - z^*a}\,|0\rangle = e^{-|z|^2/2}\,e^{za^\dagger}\,e^{-z^*a}\,|0\rangle\\ &=\; e^{-|z|^2/2}\,e^{za^\dagger}\,|0\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n a^{\dagger\,n}}{n!}\,|0\rangle\\ &=\; e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle\;. \end{aligned} \tag{20}$$

In this expression we have introduced $z = (q+ip)/\sqrt{2}$, $a = (Q+iP)/\sqrt{2}$, from which it follows that $[a,a^\dagger] = 1$ and $a\,|0\rangle = 0$, and finally the normalized states $|n\rangle \equiv$

$a^{\dagger\, n}\,|0\rangle/\sqrt{n!}$ for all $n \geq 0$. These are the familiar canonical coherent states; however, since we will generalize expressions well beyond these particular states, we shall not focus on the complex parametrization which is generally limited to this special family of coherent states.

Returning to the phase space notation, the Husimi phase-space representation of B is given by

$$\begin{aligned} B_H[p,q] &\equiv \langle p,q|\,B(P,Q)\,|p,q\rangle \\ &= \langle 0|\,B(P+p,Q+q)\,|0\rangle\ ; \end{aligned} \tag{21}$$

and the operator representation given by

$$A \equiv \int A_{G-S}[p,q]\,|p,q\rangle\,\langle p,q|\,\frac{dp\,dq}{2\pi}\ , \tag{22}$$

involves the Glauber-Sudarshan weight function $A_{G-S}[p,q]$ in a weighted integral over the one dimensional projection operators $|p,q\rangle\,\langle p,q|$. It is clear that the two symbols $A_{G-S}[p,q]$ and $B_H[p,q]$ are related to their respective operators (i.e., A and B) in very different ways. Nevertheless, one has the important relation

$$\begin{aligned} \mathrm{Tr}\,(A^\dagger\,B) &= \int A_{G-S}[p,q]^*\,\mathrm{Tr}\,(|p,q\rangle\,\langle p,q|\,B)\,\frac{dp\,dq}{2\pi} \\ &= \int A_{G-S}[p,q]^*\,B_H[p,q]\,\frac{dp\,dq}{2\pi}\ , \end{aligned} \tag{23}$$

and it is in this sense that we refer to these two distinct representations as dual to one another.

Despite their very different appearances, there is a fairly close connection between the initial, symmetric Weyl representations and the latter, asymmetric Husimi – Glauber-Sudarshan representations. That relationship is most easily seen in the Fourier transforms of the given symbols. First, let us start from the symmetric Weyl expression (11) and modify that expression in the following way:

$$\begin{aligned} &\mathrm{Tr}\,(A^\dagger\,B) \\ &= \int \{e^{(k^2+x^2)/4}\,\tilde{A}[k,x]\}^*\,\{e^{-(k^2+x^2)/4}\,\tilde{B}[k,x]\}\,\frac{dk\,dx}{2\pi} \\ &\equiv \int \tilde{A}_{G-S}[k,x]^*\,\tilde{B}_H[k,x]\,\frac{dk\,dx}{2\pi}\ . \end{aligned} \tag{24}$$

In (24) we have introduced

$$\begin{aligned} \tilde{A}_{G-S}[k,x] &= e^{(k^2+x^2)/4}\,\tilde{A}[k,x] \\ &= \int e^{i(px-qk)}\,A_{G-S}[p,q]\,dp\,dq/2\pi\ , \end{aligned} \tag{25}$$

and

$$\begin{aligned} \tilde{B}_H[k,x] &= e^{-(k^2+x^2)/4}\,\tilde{B}[k,x] \\ &= \int e^{i(px-qk)}\,B_H[p,q]\,dp\,dq/2\pi\ . \end{aligned} \tag{26}$$

This connection between the Weyl and the Husimi – Glauber-Sudarshan representations has been known for some time (see, e.g., Ref. [8], p. 185), and we do not repeat a proof here; it will be implicitly reestablished in what follows.

A direct connection also exists between the Husimi and the Glauber-Sudarshan representations as well. In particular, we can transform Eq. (22) to read

$$\begin{aligned} A_H[r,s] &= \int A_{G-S}[p,q]\,|\langle p,q|r,s\rangle|^2\,\frac{dp\,dq}{2\pi} \\ &= \int A_{G-S}[p,q]\,e^{-[(r-p)^2+(s-q)^2]/2}\,\frac{dp\,dq}{2\pi}\ . \end{aligned} \tag{27}$$

The functional form of this equation as a convolution reflects the connection through multiplication in the Fourier space.

It is clear from the connections above that the Husimi symbol $B_H[p,q]$ is generally "smoother" than the corresponding Weyl symbol $B[p,q]$, while the Glauber-Sudarshan symbol $A_{G-S}[p,q]$ is generally "rougher" than the corresponding Weyl symbol $A[p,q]$. In any asymmetric treatment of the representations this dichotomy is inevitable. In particular, it is known (see Ref. [8], p. 183) that for bounded operators, for example, the Glauber-Sudarshan symbol $A_{G-S}[p,q] \in \mathcal{Z}'(\mathbb{R}^2)$, which is the Fourier transform of the more familiar space of distributions in two dimensions, $\mathcal{D}'(\mathbb{R}^2)$. The use of suitable distributions for phase-space symbols is perfectly acceptable provided that they are always paired with dual symbols that lie in the appropriate test function space.

III. GENERALIZED ASYMMETRIC PHASE SPACE REPRESENTATIONS

We now come to the main topic of this paper, namely the introduction of a large class of asymmetric phase space representations. In so doing, we shall see how this analysis incorporates and generalizes the familiar asymmetric example discussed in the previous section.

Let us introduce a nonvanishing operator σ which we require to be a trace-class operator, i.e., we require that $0 < \mathrm{Tr}(\sqrt{\sigma^\dagger\sigma}) < \infty$. Such operators have the generic form given by

$$\sigma = \sum_{l=0}^{\infty} c_l\,|b_l\rangle\,\langle a_l|\ , \tag{28}$$

where $\{|a_l\rangle\}_{l=0}^{\infty}$ and $\{|b_l\rangle\}_{l=0}^{\infty}$ denote two, possibly identical, complete orthonormal sets of vectors, and the coefficients $\{c_l\}_{l=0}^{\infty}$ satisfy the condition

$$\mathrm{Tr}(\sqrt{\sigma^\dagger\sigma}) = \sum_{l=0}^{\infty} |c_l| < \infty\ . \tag{29}$$

If $\sigma^\dagger = \sigma$, we may choose $|b_l\rangle = |a_l\rangle$, and the coefficients c_l as real for all l; however, it is not required that σ be Hermitian.

We shall have need of the function $\mathrm{Tr}(U[k,x]\sigma)$ defined for all (k,x) in phase space. Observe that because σ is trace class, this function is *continuous*. For now, we insist that the expression

$$\mathrm{Tr}(U[k,x]\,\sigma) \neq 0 \;, \tag{30}$$

for all $(k,x) \in \mathbb{R}^2$; later, we briefly discuss a relaxation of this condition, but we will always require that $\mathrm{Tr}(\sigma) \neq 0$. The operator σ will allow us to generalize the discussion of the previous section. As an advance notice we point out that if we make the special choice that

$$\sigma = |0\rangle\langle 0| \;, \tag{31}$$

then the general discussion that follows refers to the case of the Husimi – Glauber-Sudarshan dual pairs.

We begin again with the symmetric expression for $\mathrm{Tr}(A^\dagger B)$ given by (11) which we modify so that

$$\begin{aligned}
&\mathrm{Tr}(A^\dagger B)\\
&= \int \frac{\tilde{A}[k,x]^*}{\mathrm{Tr}(U[k,x]\sigma)}\,\{\mathrm{Tr}(U[k,x]\sigma)\tilde{B}[k,x]\}\,\frac{dk\,dx}{2\pi}\\
&= \int \{\frac{\tilde{A}[k,x]}{\mathrm{Tr}(U[k,x]^\dagger\sigma^\dagger)}\}^*\,\{\mathrm{Tr}(U[k,x]\sigma)\tilde{B}[k,x]\}\,\frac{dk\,dx}{2\pi}\\
&\equiv \int \tilde{A}_{-\sigma}[k,x]^*\,\tilde{B}_\sigma[k,x]\,\frac{dk\,dx}{2\pi}\\
&\equiv \int A_{-\sigma}[p,q]^*\,B_\sigma[p,q]\,\frac{dp\,dq}{2\pi}\;.
\end{aligned} \tag{32}$$

In the final line we have introduced the Fourier transform of the symbols in the line above. Our task is to find alternative expressions involving the symbols $A_{-\sigma}[p,q]$ and $B_\sigma[p,q]$ directly in their own space of definition rather than implicitly through a Fourier transformation.

We begin first with the symbol $B_\sigma[p,q]$. In particular, we note that

$$\begin{aligned}
B_\sigma[p,q] &= \int e^{i(kq-xp)}\,\mathrm{Tr}(U[k,x]\sigma)\tilde{B}[k,x]\,\frac{dk\,dx}{2\pi}\\
&= \int \mathrm{Tr}(U[p,q]^\dagger\,U[k,x]\,U[p,q]\sigma)\,\mathrm{Tr}(U[k,x]^\dagger B)\,\frac{dk\,dx}{2\pi}\\
&= \int \mathrm{Tr}(U[k,x]\,U[p,q]\,\sigma\,U[p,q]^\dagger)\,\mathrm{Tr}(U[k,x]^\dagger B)\,\frac{dk\,dx}{2\pi}\\
&= \mathrm{Tr}(U[p,q]\,\sigma\,U[p,q]^\dagger B)\;,
\end{aligned} \tag{33}$$

where in the middle line we have used the Weyl form of the commutation relations Eq. (2), and in the last line we have used Eq. (5), which has led us to the desired expression for $B_\sigma[p,q]$. This expression is the sought for generalization of the Husimi representation; indeed, if $\sigma = |0\rangle\langle 0|$ it follows immediately that

$$\begin{aligned}
B_\sigma[p,q] &= \mathrm{Tr}(U[p,q]\,|0\rangle\langle 0|U[p,q]^\dagger B)\\
&= \langle p,q|\,B\,|p,q\rangle = B_H[p,q]\;.
\end{aligned} \tag{34}$$

For general σ, to find the expression for $A_{-\sigma}[p,q]$ we appeal to the relation

$$\begin{aligned}
\mathrm{Tr}(A^\dagger B) &= \int A_{-\sigma}[p,q]^*\,B_\sigma[p,q]\frac{dp\,dq}{2\pi}\\
&= \int A_{-\sigma}[p,q]^*\mathrm{Tr}(U[p,q]\,\sigma\,U[p,q]^\dagger B)\frac{dp\,dq}{2\pi}\;,
\end{aligned} \tag{35}$$

an equation which, thanks to its validity for all B of the form $B = |\phi\rangle\langle\psi|$ for arbitrary $|\phi\rangle$ and $|\psi\rangle$ in the Hilbert space, carries the important implication that

$$A^\dagger \equiv \int A_{-\sigma}[p,q]^*\,U[p,q]\,\sigma\,U[p,q]^\dagger\,\frac{dp\,dq}{2\pi}\;, \tag{36}$$

or if we take the Hermitian adjoint that

$$A \equiv \int A_{-\sigma}[p,q]\,U[p,q]\,\sigma^\dagger\,U[p,q]^\dagger\,\frac{dp\,dq}{2\pi}\;. \tag{37}$$

Observe that this equation implies a very general operator representation as a linear superposition of basic operators given by $U[p,q]\sigma^\dagger\,U[p,q]^\dagger$, for a general choice of σ that satisfies the conditions given initially.

Equation (37) for A is the sought for generalization of the Glauber-Sudarshan representation; indeed, if $\sigma = |0\rangle\langle 0|$, it follows immediately that

$$\begin{aligned}
A &= \int A_{-\sigma}[p,q]\,U[p,q]\,|0\rangle\langle 0|U[p,q]^\dagger\,\frac{dp\,dq}{2\pi}\\
&= \int A_{-\sigma}[p,q]\;|p,q\rangle\langle p,q|\,\frac{dp\,dq}{2\pi} \qquad (38)\\
&= \int A_{G-S}[p,q]\;|p,q\rangle\langle p,q|\,\frac{dp\,dq}{2\pi}\;. \qquad (39)
\end{aligned}$$

Once again there is a direct connection between the generalization of the Husimi representation, $A_\sigma[p,q]$, and the generalization of the Glauber-Sudarshan representation, $A_{-\sigma}[p,q]$. In particular, it follows from (37) that

$$\begin{aligned}
&A_\sigma[r,s] =\\
&= \int A_{-\sigma}[p,q]\,\mathrm{Tr}(U[r,s]\sigma U[r,s]^\dagger U[p,q]\sigma^\dagger U[p,q]^\dagger)\,\frac{dp\,dq}{2\pi}\\
&= \int A_{-\sigma}[p,q]\\
&\qquad\times[\mathrm{Tr}(U[r-p,q-s]\sigma U[r-p,q-s]^\dagger\sigma^\dagger)\,\frac{dp\,dq}{2\pi}\;.
\end{aligned} \tag{40}$$

Again, this equation is a convolution, which just reflects the multiplicative connection between these two symbols in the Fourier space. Note that the convolution kernel in (40) is generally complex unless $\sigma^\dagger = \sigma$; c.f., Eq. (27).

IV. EXAMPLES OF GENERALIZED PHASE SPACE OPERATOR REPRESENTATIONS

In this section we offer a sample of the generalization offered by our formalism. In particular, let us choose a thermal density matrix

$$\sigma = Z\,e^{-\beta N}\;, \tag{41}$$

where $\beta > 0$, $N = a^\dagger a$ is the number operator, with spectrum $\{0,1,2,3,\ldots\}$, and $Z = 1 - e^{-\beta}$ normalizes

σ so that $\mathrm{Tr}(\sigma)=1$. Observe that if we take a limit in which $\beta\to\infty$, then $\sigma\to|0\rangle\langle 0|$ appropriate to the standard case. For convenience, we label this example simply by the parameter β.

It follows first that

$$\mathrm{Tr}(U[k,x]\sigma)=Z\,\mathrm{Tr}(U[k,x]e^{-\beta N})=e^{-\theta(k^2+x^2)/4}, \quad (42)$$

where

$$\theta\equiv\frac{1+e^{-\beta}}{1-e^{-\beta}}\;. \quad (43)$$

Clearly, $\theta>1$, and $\theta\to 1$ as $\beta\to\infty$. Furthermore,

$$\begin{aligned}B_\beta[p,q]&=Z\,\mathrm{Tr}(U[p,q]e^{-\beta N}U[p,q]^\dagger B)\\&=(1-e^{-\beta})\sum_{n=0}^{\infty}e^{-\beta n}\langle n|U[p,q]^\dagger B(P,Q)U[p,q]|n\rangle\\&=(1-e^{-\beta})\sum_{n=0}^{\infty}e^{-\beta n}\langle n|B(P+p,Q+q)|n\rangle\;. \quad (44)\end{aligned}$$

This expression admits alternative forms as well. Based on the relation

$$\tilde{B}_\beta[k,x]=e^{-\theta(k^2+x^2)/4}\,\tilde{B}[k,x]\;, \quad (45)$$

it follows that

$$B_\beta[p,q]=\int e^{-(1/\theta)[(p-p')^2+(q-q')^2]}\,\frac{B[p',q']}{\pi\theta}\,\frac{dp'dq'}{2\pi}\;. \quad (46)$$

Still another connection is given by

$$\begin{aligned}B_\beta[p,q]&=\int e^{i(kq-xp)}\,e^{-\theta(k^2+x^2)/4}\,\tilde{B}[k,x]\,\frac{dk\,dx}{2\pi}\\&=e^{(\theta/4)(\partial^2/\partial p^2+\partial^2/\partial q^2)}\int e^{i(kq-xp)}\,\tilde{B}[k,x]\,\frac{dk\,dx}{2\pi}\\&=e^{(\theta/4)(\partial^2/\partial p^2+\partial^2/\partial q^2)}\,B[p,q]\;. \quad (47)\end{aligned}$$

Let us turn our attention to the dual representation associated with $A_{-\beta}[p,q]$. In this case, we focus on the operator representation given by

$$\begin{aligned}A&=(1-e^{-\beta})\int A_{-\beta}[p,q]\,U[p,q]e^{-\beta N}U[p,q]^\dagger\,\frac{dp\,dq}{2\pi}\\&=(1-e^{-\beta})\sum_{n=0}^{\infty}e^{-\beta n}\int A_{-\beta}[p,q]\\&\qquad\times|p,q;n\rangle\langle p,q;n|\frac{dp\,dq}{2\pi}\;, \quad (48)\end{aligned}$$

where we have introduced the notation [9]

$$|p,q;n\rangle\equiv U[p,q]\,|n\rangle \quad (49)$$

for those coherent states for which the fiducial vector is $|n\rangle$. There is an alternative representation for the symbol under discussion which is given by

$$A_{-\beta}[p,q]=e^{-(\theta/4)(\partial^2/\partial p^2+\partial^2/\partial q^2)}\,A[p,q]\;, \quad (50)$$

which is particularly useful if the Weyl symbol is a polynomial.

We now turn our attention to a different but closely related example, which we distinguish by a prime ($'$). In particular, we choose

$$\sigma'\equiv Z'\,(-1)^N\,e^{-\beta N}=(1+e^{-\beta})\,e^{-(\beta+i\pi)\,N}\;. \quad (51)$$

This example also corresponds to a Hermitian choice for σ, but it is not positive definite as was our previous choice. The results of interest are easily calculated simply by an analytic extension of the parameter β to $\beta'\equiv\beta+i\pi$. In particular, it follows that

$$\mathrm{Tr}(U[k,x]\,\sigma')=e^{-\theta'(k^2+x^2)/4}\;, \quad (52)$$

where

$$\theta'=\frac{1+e^{-\beta'}}{1-e^{-\beta'}}=\frac{1-e^{-\beta}}{1+e^{-\beta}}\;. \quad (53)$$

Thus, in the present case, we have a similar rescaling factor, but this time, the parameter $\theta'<1$, with again the limit that $\theta'\to 1$ as $\beta\to\infty$.

Turning attention to the relevant symbols, we first see that

$$\begin{aligned}B_{\beta'}[p,q]&=Z'\,\mathrm{Tr}(U[p,q]e^{-\beta' N}U[p,q]^\dagger B)\\&=(1+e^{-\beta})\sum_{n=0}^{\infty}(-1)^n\,e^{-\beta n}\langle n|U[p,q]^\dagger B(P,Q)U[p,q]|n\rangle\\&=(1+e^{-\beta})\sum_{n=0}^{\infty}(-1)^n\,e^{-\beta n}\langle n|B(P+p,Q+q)|n\rangle\;. \quad (54)\end{aligned}$$

This expression also admits alternative forms as well. Based on the relation

$$\tilde{B}_{\beta'}[k,x]=e^{-\theta'(k^2+x^2)/4}\,\tilde{B}[k,x]\;, \quad (55)$$

it follows that

$$B_{\beta'}[p,q]=\int e^{-(1/\theta')[(p-p')^2+(q-q')^2]}\,\frac{B[p',q']}{\pi\theta'}\,\frac{dp'dq'}{2\pi}\;. \quad (56)$$

Still another connection is given by

$$B_{\beta'}[p,q]=e^{(\theta'/4)(\partial^2/\partial p^2+\partial^2/\partial q^2)}\,B[p,q]\;. \quad (57)$$

Let us again turn our attention to the dual representation associated with $A_{-\beta'}[p,q]$. In this case, we focus on the operator representation given by

$$\begin{aligned}A&=(1-e^{-\beta'})\int A_{-\beta'}[p,q]\,U[p,q]e^{-\beta' N}U[p,q]^\dagger\frac{dp\,dq}{2\pi}\\&=(1+e^{-\beta})\sum_{n=0}^{\infty}(-1)^n e^{-\beta n}\int A_{-\beta'}[p,q]\\&\qquad\times|p,q;n\rangle\,\langle p,q;n|\frac{dp\,dq}{2\pi}\;. \quad (58)\end{aligned}$$

There is another representation for the symbol under discussion given by

$$A_{-\beta'}[p,q] = e^{-(\theta'/4)(\partial^2/\partial p^2+\partial^2/\partial q^2)}\,A[p,q]\;, \qquad (59)$$

which is particularly useful, once again, if the Weyl symbol is a polynomial.

It is of interest to observe, for the thermal choice of σ (or one with a complex temperature such as σ') discussed in this section, that every bounded operator A may be represented for any choice of β by weights $A_{-\beta}[p,q]$ that are elements of $\mathcal{Z}'(\mathbb{R}^2)$, just as was the case for the Glauber-Sudarshan representation. This holds because if we multiply a distribution in $\mathcal{D}'(\mathbb{R}^2)$ by an expression of the form $\exp[\theta(k^2+x^2)/4]$, it remains a distribution in $\mathcal{D}'(\mathbb{R}^2)$ because multiplication with such a factor leaves the test function space $\mathcal{D}(\mathbb{R}^2)$ invariant.

The example discussed above has some similarity with one treated by Cahill and Glauber [10]. In their work, they interpret the additional factor as arising from a reordering of the basic Weyl operator, i.e., a smooth transition from normal ordered to anti-normal ordered. On the other hand, we interpret a similar factor as coming from an alternative definition of the associated symbol.

V. FURTHER EXAMPLES AND GENERALIZATIONS

A. Additional examples

For a fixed, nonzero point (r,s) in phase space, consider the non-Hermitian example for which

$$\sigma = (1-e^{-\beta})\,U[r,s]^\dagger\,e^{-\beta N}\;. \qquad (60)$$

In this case

$$\begin{aligned}&\mathrm{Tr}\,(U[k,x]\sigma) = (1-e^{-\beta})\,\mathrm{Tr}\,(U[k,x]\,U[r,s]^\dagger\,e^{-\beta N})\\ &= (1-e^{-\beta})\,e^{-i(ks-xr)/2}\,\mathrm{Tr}\,(U[k-r,x-s]\,e^{-\beta N})\\ &= e^{-i(ks-xr)/2}\,e^{-\theta[(k-r)^2+(x-s)^2]/4}\;, \end{aligned}\qquad (61)$$

where

$$\theta \equiv \frac{1+e^{-\beta}}{1-e^{-\beta}} \qquad (62)$$

as before. With this example, we find that

$$\begin{aligned}&\tilde{B}_\sigma[k,x] =\\ &e^{-i(ks-xr)/2}\,e^{-\theta[(k-r)^2+(x-s)^2]/4}\,\mathrm{Tr}\,(U[k,x]^\dagger B)\;,\end{aligned} \qquad (63)$$

which has the feature that phase space points at and near (r,s) are "emphasized" in comparison with the rest of phase space. For applications in the time-frequency domain, for example, such an emphasis may of value.

Another class of examples is offered by the following observation. First, let us make a preliminary choice of $\sigma = |n\rangle\langle n|$, where $|n\rangle$ denotes the n particle state introduced earlier. For such a choice we learn that (see, e.g., [10])

$$\begin{aligned}&\mathrm{Tr}\,(U[k,x]\,|n\rangle\langle n|) = \langle n|U[k,x]\,|n\rangle\\ &= \mathrm{L}_n(\tfrac{1}{2}(k^2+x^2))\,e^{-(k^2+x^2)/4}\;,\end{aligned} \qquad (64)$$

where $\mathrm{L}_n(y)$ denotes a Laguerre polynomial defined, as usual, by

$$\mathrm{L}_n(y) \equiv \frac{e^y}{n!}\frac{d^n}{dy^n}\,(y^n\,e^{-y})\;. \qquad (65)$$

Each Laguerre polynomial $\mathrm{L}_n(y)$ is real and has n distinct, real zeros. Thus the choice $\sigma = |n\rangle\langle n|$ violates the basic postulate that $\mathrm{Tr}\,(U[k,x]\sigma)\neq 0$ for all $(k,x)\in\mathbb{R}^2$ whenever $n>0$. However, we can overcome this problem by choosing instead a non-Hermitian example, such as

$$\sigma = R\,|n\rangle\langle n| + iT\,|m\rangle\langle m|\;, \qquad (66)$$

where $m\neq n$, and R and T are two real, nonzero factors that reflect the relative weight of the two contributions. As a consequence, we are led to

$$\begin{aligned}\mathrm{Tr}\,(U[k,x]\sigma) = [R\,\mathrm{L}_n(\tfrac{1}{2}(k^2+x^2))&\\ +\,iT\,\mathrm{L}_m(\tfrac{1}{2}(k^2+x^2))]\,e^{-(k^2+x^2)/4}&\;,\end{aligned} \qquad (67)$$

which is never zero for any choice of (k,x). Further extensions along these lines are evident.

Associated with each choice of σ above is the corresponding dual representation given, as usual, by

$$A \equiv \int A_{-\sigma}[p,q]\,U[p,q]\,\sigma^\dagger\,U[p,q]^\dagger\,\frac{dpdq}{2\pi}\;, \qquad (68)$$

which we do not need to spell out in detail again.

B. Relaxation of nonvanishing criterion

Returning to a general choice of σ, we observe that so long as $\mathrm{Tr}\,(U[k,x]\sigma)$ never vanishes, we can assert that if $B_\sigma[p,q]=0$ for all $(p,q)\in\mathbb{R}^2$, then the operator $B=0$. This property follows from the fact that when $\tilde{B}[k,x]=0$, it follows that $\mathrm{Tr}\,(B^\dagger B)=\int|\tilde{B}[k,x]|^2\,dk\,dx=0$, which can only happen if $B=0$. However, if

$$\mathrm{Tr}\,(U[k_o,x_o]\sigma)=0 \qquad (69)$$

for some phase space point (k_o,x_o), for example, then the very operator $B=U[k_o,x_o]$ would lead to the fact that $\tilde{B}[k,x]=0$, for all (k,x) in phase space, with the consequence that $B_\sigma[p,q]\equiv 0$ despite the fact that $B\neq 0$.

Turning to the dual symbols, we observe that so long as $\mathrm{Tr}\,(U[k,x]\sigma)$ never vanishes, we can formally represent every operator in the form

$$A = \int A_{-\sigma}[p,q]\,U[p,q]\,\sigma^\dagger\,U[p,q]^\dagger\,\frac{dpdq}{2\pi}, \qquad (70)$$

where the symbol

$$A_{-\sigma}[p,q] = \int e^{i(px-qk)} \frac{\tilde{A}[k,x]}{\mathrm{Tr}(U[k,x]^\dagger \sigma^\dagger)} \frac{dk\,dx}{2\pi} \tag{71}$$

is a distribution appropriate to the situation under consideration. More precisely, we can represent all bounded operators in such a fashion, and every operator can be obtained as a suitable limit from the set of bounded operators. On the other hand, if we relax the condition that $\mathrm{Tr}(U[k,x]\sigma)$ never vanishes, we loose this generality. Suppose again that $\mathrm{Tr}(U[k_o,x_o]\sigma) = 0$ for some particular point in phase space (k_o,x_o). In that case, it is strictly speaking not possible to generate the operator $U[k_o,x_o]\sigma^\dagger U[k_o,x_o]^\dagger$.

On the other hand, if one is only interested in operators that are *polynomial* in P and Q, then the support of the Weyl symbol $\tilde{A}[k,x]$ or $\tilde{B}[k,x]$ is strictly at the origin in phase space, e.g., for $\tilde{C}$ equal either $\tilde{A}$ or $\tilde{B}$,

$$\tilde{C}[k,x] = \sum_{i,j=0}^{I,J} c_{i,j} \frac{d^i}{dk^i} \frac{d^j}{dx^j} \delta(k)\,\delta(x) \;, \tag{72}$$

where $I < \infty$ and $J < \infty$. In that case, the fact that $\mathrm{Tr}(U[k,x]\sigma)$ may vanish away from the origin has no influence on the matter, and thus so long as $\mathrm{Tr}(\sigma) \neq 0$, and thus by continuity $\mathrm{Tr}(U[k,x]\sigma)$ is nonzero in an open neighborhood of the origin, *all* polynomials in P and Q can be represented in the form

$$A = \int A_{-\sigma}[p,q]\, U[p,q]\sigma^\dagger U[p,q]^\dagger \frac{dp\,dq}{2\pi} \;, \tag{73}$$

and, additionally, the symbol

$$B_\sigma[p,q] = \mathrm{Tr}(U[p,q]\sigma U[p,q]^\dagger B) \tag{74}$$

uniquely determines the operator B provided that it is a polynomial. This result generalizes a result established some time ago [11].

C. Arbitrary weight factors

It is of course possible to reweight the Weyl representation by quite arbitrary factors in the manner

$$\begin{aligned}
\mathrm{Tr}(A^\dagger B) &= \int \tilde{A}[k,x]^*\,\tilde{B}[k,x]\frac{dk\,dx}{2\pi} \\
&= \int \{(F[k,x]^*)^{-1}\,\tilde{A}[k,x]\}^*\,\{F[k,x]\,\tilde{B}[k,x]\}\frac{dk\,dx}{2\pi} \\
&\equiv \int \tilde{A}_{-F}[k,x]^*\,\tilde{B}_F[k,x]\frac{dk\,dx}{2\pi} \\
&\equiv \int A_{-F}[p,q]\,B_F[p,q]\frac{dp\,dq}{2\pi} \;,
\end{aligned} \tag{75}$$

where $F[k,x]$ is an arbitrary nonvanishing factor, and, as usual, we have introduced the Fourier transform elements in the last line. Apart from additional conditions – specifically, $F(0,x) = 1 = F(k,0)$ – this is the procedure used by Cohen [2]. However, a general factor $F[k,x]$ cannot normally be represented as $\mathrm{Tr}(U[k,x]\sigma)$ for some σ. For example, the function $\mathrm{Tr}(U[k,x]\sigma)$ is necessarily continuous and bounded by $|\mathrm{Tr}(U[k,x]\sigma)| \leq \mathrm{Tr}(\sqrt{\sigma^\dagger\sigma})$, the trace class norm of σ. However, more to the point, the introduction of a general expression such as $F[k,x]$ would normally introduce elements outside the Hilbert space and its operators that we have so far consistently stayed within. Such an extension may of course be considered, but the non-operator nature in the extension being made should be appreciated and accepted.

Of course, the investigation of quasi-distributions continues unabated. As one comparatively recent example of such investigations, we cite the work of [12].

D. Non-canonical generalizations

Representations of Hilbert space operators in the manner of the Weyl representation may be carried out for a great variety of groups. In addition, since coherent states may be defined for other groups, analogs of the Husimi and dual Glauber-Sudarshan representations exist in such cases as well, and these have often been discussed in the literature. Consequently, it follows that asymmetric representations of various forms, analogous to those presented in this paper for the Weyl group, can be introduced for other groups, e.g., the groups SU(2) and SU(1, 1), as well.

VI. CONCLUSIONS

Quite naturally, an increase in the family of representations of various systems offers new ways to study such systems. A general formulation of states and observables must lead to a symmetric, abstract description; but for *specific* systems, where the states of interest may be "better behaved" than the observables, it may be useful to take advantage of that distinction with an asymmetric representation pair. Likewise, within the realm of time-frequency analysis, signals and their analyzers may have very different characteristics that also suggest the utility of an asymmetric pair. Moreover, to minimize any extraneous elements in such asymmetric representations, it is appropriate that the representations in question be formulated within the initial abstract formalism, a point of view developed in the present paper.

Although it is difficult to predict in advance just which asymmetric representation pairs may prove useful, there may well arise special situations for which certain asymmetric representations prove useful – just as was the case in quantum optics for the Husimi – Glauber-Sudarshan representation pair.

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